

# Proton radius, bound state QED and the nonlocality of the electromagnetic interaction.

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## Abstract

The result of a recent measurement of the size of the proton [R. Pohl et al., Nature **466**, 213] performed on the base of the muonic hydrogen spectroscopy turned out to be significantly different, by five standard deviations, from the results derived from the atomic hydrogen spectroscopy. This large discrepancy could come from the calculations of the Lamb shift in atomic hydrogen and muonic hydrogen. Here we show that there is a gap in the standard bound-state QED that may be the source of the discrepancy. This gap originates in the fact that within the framework of this theory the QED corrections are described in terms of the respective Green functions. The character of the time evolution of a system which should manifest itself in the general definition of bound states as stationary states of the system cannot be described in terms of the Green functions. We present a consistent way of solving the bound-state problem in QED starting from the condition of stationarity of the bound states. Formulae for the energies and the vectors of the states of one-electron (muon) atoms derived in this way indicate that the standard bound-state QED does not obey the exact description of the atomic states and, as a result, the Lamb shift obtained in its framework should be supplemented by an additional "dynamical" energy shift. It is shown that in this shift natural nonlocality of the electromagnetic interaction that in describing the S matrix and the Green functions is hidden in the renormalization procedure manifest itself explicitly.

*Keywords:* Proton radius, bound states QED, nonlocal interactions

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## Introduction

The development of the theory of quantum electrodynamics was stimulated by high-precision hydrogen spectroscopy performed by Lamb and Retherford in 1947 that showed a small splitting between the  $2S_{1/2}$  and  $2P_{1/2}$  states known as the Lamb shift. Despite the theory of QED is the most well-tested, accurate, and successful theory in physics its mathematical foundation is not secure because of the still unsolved problem of the ultraviolet (UV) divergences. The renormalization theory that in QED is used for removing these divergences actually allows one to pass over this problem but not to solve them. Richard Feynman said in this connection: "I think that the renormalization theory is

simply a way to sweep the difficulties of the divergences of electrodynamics under the rug" [1]. As is well known, in QED the ultraviolet divergences can be removed from the S matrix and the Green functions, but cannot be removed from quantities characterizing the time evolution of quantum systems, since regularization and renormalization of the scattering matrix leads to the situation in which divergent terms automatically appear in the Hamiltonian and hence in the Schrödinger equation [2]. From this point of view the results of the recent muonic hydrogen Lamb shift experiment [3] at PSI that have led to the proton charge radius significantly different, by five standard deviations, from the that derived from hydrogen atom spectroscopy [4–7] value seem not surprising. As Jeff Flowers of Britain's National Physical Laboratory wrote in Ref. [8], if these experimental discrepancy is confirmed then "high-accuracy work such as that by Pohl and colleagues, not the high-energy collisions of giant accelerators, may have seen beyond the standard model of particle physics".

Here we show that the discrepancy may originate from the fact that the theory of QED fails in describing the time evolution of a system. For this reason one is forced to describe the bound states of QED systems in terms of the S matrix or the Green functions. However, in general bound states should be determined as stationary states which evolve in time according to the law:

$$|\Psi_n(t)\rangle = e^{-iE_n t} |\Psi_n(0)\rangle. \quad (1)$$

If in QED the Hamiltonian, i.e. the operator of the total energy, were well defined, then from this definition it should immediately follow that the vectors of the stationary states and their energies are eigenvectors of the Hamiltonian  $H|\Psi_n\rangle = E_n|\Psi_n\rangle$ . However, this is not the case. The time evolution of QED systems remains uncertain even after renormalization. Actually this demonstrates the limitation of the existing theory of QED. The problem could be passed over, provided the bound states of composite systems in QED could be described in terms of the renormalizable S matrix and the Green functions. However, the discrepancy between the proton radii deduced from atomic hydrogen spectroscopy and muonic hydrogen spectroscopy shows that this may not be the case: the QED corrections that played an important role in the determination of these radii had been calculated by using the standard methods of the bound state QED in which the energy levels of composite systems are determined by the positions of poles of the respective Green functions. This makes it necessary to investigate the problem more precisely. Since locality has been argued to be the main cause of infinities in QED, it seems natural to resolve this problem by introducing a nonlocal form factor into the interaction Hamiltonian density. However, it turned out that such an introduction of a nonlocal form factor results in the loss of covariance. The origin of this is the fact that the Schrödinger equation is local in time, and the interaction Hamiltonian describes an instantaneous interaction. But in relativistic quantum theory, if we spread the interaction in space, then we should spread it in time as well. Thus, for the introduction of nonlocality in the theory to be intrinsically consistent, a way is needed of solving the evolution problem in the case when the dynamics

in a system is generated by a nonlocal-in-time interaction. In Ref. [9] it has been shown that actually the Schrödinger equation is not the only dynamical equation consistent with the current concept of quantum physics, and the most general dynamical equation (GDE) consistent with these principles has been derived. Being equivalent to the Schrödinger equation in the case of instantaneous interaction, this generalized dynamical equation allows one to generalize the dynamics to the case where the dynamics is governed by a nonlocal-in-time interaction.

The GDE is shown to provide a consistent way of solving the bound-state problem starting from the law (1) determining bound states. In this way formulae are derived that determine the energies and the vectors of the states of one-electron (muon) atoms. The values of the atomic energy levels determined by these formulae differ from the values determined by the positions of the poles of the Green function of the electron in the Coulomb field. This difference that will be referred to as the dynamical shift is the reflection of the fact that there is not the exact correspondence between the atomic states and the poles of the Green function. The dynamical shift is shown to get the contributions from the processes, in which the nonlocal nature of the electromagnetic interaction that in describing the S matrix and the Green function is hidden in the renormalization procedure manifests itself explicitly.

## 1. Quantum mechanical rules and the generalized dynamical equation

Paraphrasing Steven Weinberg [10] the recent situation in QED may be characterized as follows. If it turned out some QED system could not be described by the theory of QED, it would be a sensation. However, if it turned out that the system did not obey the rules of quantum mechanics and relativity, it would be a cataclysm. Of course, the question is raised, in this connection, what are the basic quantum mechanical rules that must be satisfied in any theory. In the Feynman's book [11] where a minimal set of physical principles that must be satisfied in any theory of fields and particles is analyzed, the only quantum mechanical principle included in this set is the principle of the superposition of probability amplitudes. This principle formulated as the result of analysis of the phenomenon of quantum interference [12] reads as follows.

The probability of an event is the absolute square of a complex number called the probability amplitude. The probability amplitude of an event which can happen in several different ways is a sum of the probability amplitudes for each of these ways.

In the Feynman formulation of quantum theory [12, 13] this principle is used as a basic postulate. This postulate provides the general rule prescribing how to calculate probabilities in quantum theory, and can be used in different ways depending on the choice of the class of alternative ways in which events can happen. In the Feynman formalism the processes associated with completely specified paths of particles in space-time are used as such alternatives. The contribution from a single path is postulated to be an exponential whose (imaginary) phase is the classical action (in units of  $\hbar$ ) to the path in question. Thus

in the case of such a choice of the class of alternatives the contribution of each of the alternative way must be specified from the every beginning. However, very surprisingly it has turned out [9] that there is a much more general class of alternatives whose contributions need not to be postulated: it is enough to know a priori only the contributions from the alternative processes associated with a fundamental interaction while the contributions from other alternatives are determined by the requirement of the conservation of probability. This class of alternatives consists of the processes with completely specified instants of the beginning and end of interaction in a quantum system. With such a class of alternatives, the superposition principle allows one to represent the probability amplitude of finding a quantum system in the state  $|\Psi_2\rangle$  at time  $t$ , if at time  $t_0$  it was in the state  $|\Psi_1\rangle$ , in the form [9]

$$\langle\Psi_2|U(t,t_0)|\Psi_1\rangle = \langle\Psi_2|\Psi_1\rangle + \int_{t_0}^t dt_2 \int_{t_0}^{t_2} dt_1 \langle\Psi_2|\tilde{S}(t_2,t_1)|\Psi_1\rangle. \quad (2)$$

Here  $\langle\Psi_2|\tilde{S}(t_2,t_1)|\Psi_1\rangle$  is the probability amplitude that the interaction in the system begins in the time interval  $(t_1, t_1 + dt_1)$  and ends in the time interval  $(t_2, t_2 + dt_2)$ , and after the end of the interaction the system will be found in the state  $|\Psi_2\rangle$  (we use the interaction picture), if before the beginning of the interaction the system was in the state  $|\Psi_1\rangle$ . By using the operator formalism, one can represent amplitudes  $\langle\Psi_2|U(t,t_0)|\Psi_1\rangle$  by the matrix elements of the unitary evolution operator  $U(t,t_0)$  in the interaction picture. The operator  $\tilde{S}(t_2,t_1)$  defined in the same way represents the contribution to the evolution operator from the process in which the interaction in the system begins at time  $t_1$  and ends at time  $t_2$ . As has been shown in Ref. [9], for the evolution operator in the form (2) to be unitary for any  $t$  and  $t_0$  the operator  $\tilde{S}(t_2,t_1)$  must satisfy the equation

$$(t_2 - t_1)\tilde{S}(t_2,t_1) = \int_{t_1}^{t_2} dt_4 \int_{t_1}^{t_4} dt_3 (t_4 - t_3)\tilde{S}(t_2,t_4)\tilde{S}(t_3,t_1). \quad (3)$$

A remarkable feature of this equation is that it works as a recurrence relation and allows one to obtain the operators  $\tilde{S}(t_2,t_1)$  for any  $t_1$  and  $t_2$ , if  $\tilde{S}(t'_2,t'_1)$  corresponding to infinitesimal duration times  $\tau = t'_2 - t'_1$  of interaction are known. It is natural to assume that most of the contribution to the evolution operator in the limit  $t_2 \rightarrow t_1$  comes from the processes associated with the fundamental interaction in the system under study. Denoting this contribution by  $H_{int}(t_2,t_1)$  we can write

$$\tilde{S}(t_2,t_1) \xrightarrow{t_2 \rightarrow t_1} H_{int}(t_2,t_1) + O(\tau^\epsilon), \quad (4)$$

where  $\tau = t_2 - t_1$ . The parameter  $\epsilon$  is determined by demanding that  $H_{int}(t_2,t_1)$  called the generalized interaction operator must be so close to the solution of Eq. (3) in the limit  $t_2 \rightarrow t_1$  that this equation has a unique solution having the

behavior (4) near the point  $t_2 = t_1$ . Actually, Eqs. (2) and (3) represent the quantum mechanical rules that must be obtained by any physical theory. Equation (2) is the representation of the probability amplitude  $\langle \Psi_2 | U(t, t_0) | \Psi_1 \rangle$  as a sum of the contributions from all alternative ways in which the event can happen for the chosen class of alternatives, and Eq. (3) is simply the unitarity condition in terms of these contributions. The wonderful feature of this rule is that it directly leads to the dynamical equation. If  $H_{int}(t_2, t_1)$  is specified, the "unitarity condition" (3) allows one to determine  $\tilde{S}(t_2, t_1)$  for any  $t_1$  and  $t_2$  and hence to construct the evolution operator. Thus, being supplemented by the boundary condition (4), the relationship (3) becomes the equation of motion for states of a quantum system. It is important that this equation is universal, and the specific features of a theory describing the dynamics of quantum systems manifest themselves only in the boundary condition (4) and in the Hilbert space with which the theory is dealing. In the case when the quantum field theory is required for the description of physical processes the Hilbert space must be chosen in the form of the Fock space, and correspondingly the interaction operator  $H_{int}(t_2, t_1)$  should be constructed in terms of the field operators. In the case when the interaction operator is of the form

$$H_{int}(t_2, t_1) = -2\pi i \delta(t_2 - t_1) H_I(t_1), \quad (5)$$

i.e., the fundamental interaction is instantaneous, the generalized dynamical equation (3) turns out to be equivalent [9] to the Schrödinger equation with the interaction operator  $H_I(t_1)$ . A quantum field theory must also obey the relativity. This condition is satisfied in the case when the interaction Hamiltonian density  $\mathcal{H}(x)$  related to the interaction Hamiltonian  $H_I(t_1)$  by the equation  $H_I(t) = \int d^3x \mathcal{H}_I(t=0, \mathbf{x})$  is local, i.e. the interaction is local both in time and in space. But this locality leads to the UV divergences. At the same time the interaction operator of the form (5) is the specific case of operators allowed by Eq. (4). In general the interaction operators describe the interaction spread both in space and in time. In this case the dynamics of a quantum system is determined by the behaviour of the interaction operator in the Schrödinger picture,  $\mathcal{H}_{int}^{(s)}(x)$ , in the limit of infinitesimal duration times  $\tau$  of the interaction (in the local case the dynamics is determined by the interaction operator specified at the point  $\tau = 0$ ). Thus there is no finite scale of nonlocality in the case of such nonlocal interaction! In order to obey these rules the operator describing the time evolution in QED should be of the form (2) with  $\tilde{S}(t_2, t_1)$  satisfying Eq. (3), and hence the fundamental QED interaction should be described by one of the nonlocal interaction operators allowed by Eq. (4) and obeying the rule of relativity. Thus the problem of the UV divergences is reduced to the problem of finding the form of this operator.

Formally Eq. (2) involves processes associated with vacuum-vacuum transitions. Correspondingly the matrix element  $\langle \Psi_2 | U(t, 0) | \Psi_1 \rangle$  of the evolution operator defined by Eq. (2) is the product of the probability amplitude  $\langle \Psi_2 | U_{ph}(t, 0) | \Psi_1 \rangle$  of the physical event under study and the probability ampli-

tude of the vacuum-vacuum transition:

$$\langle \Psi_2 | U(t, 0) | \Psi_1 \rangle = \langle \Psi_2 | U_{ph}(t, 0) | \Psi_1 \rangle \langle 0 | U(t, 0) | 0 \rangle.$$

This one-loop contribution is proportional to the space-volume  $V$  because of the translational invariance. For the same reason the contribution from the two-loop vacuum process is proportional to  $V^2$  and so on. The amplitude  $\langle \Psi_2 | U_{ph}(t, 0) | \Psi_1 \rangle$ , of course, is a sum of contributions from the process with the specified moments of the beginning and end of the interaction

$$\langle \Psi_2 | U_{ph}(t, 0) | \Psi_1 \rangle = \langle \Psi_2 | \Psi_1 \rangle + \int_{t_0}^t dt_2 \int_{t_0}^{t_2} dt_1 \langle \Psi_2 | \tilde{S}_{ph}(t_2, t_1) | \Psi_1 \rangle$$

However, in this case the processes of the interaction associated with the vacuum-vacuum transitions must not be taken into account. Correspondingly,  $\tilde{S}_{ph}(t_2, t_1)$  must satisfy Eq. (3), from the right-hand part of which one has to remove the terms proportional to  $V^n$ , i.e., the terms associated with the processes which involve the vacuum-vacuum transitions:

$$(t_2 - t_1) \tilde{S}_{ph}(t_2, t_1) = \int_{t_1}^{t_2} dt_4 \int_{t_1}^{t_4} dt_3 (t_4 - t_3) \tilde{S}_{ph}(t_2, t_4) \tilde{S}_{ph}(t_3, t_1) - c.t.$$

where *c.t.* stands for counter terms proportional to  $V^n$ .

In the further discussion we will use the general notation  $\langle \Psi_2 | U(t, 0) | \Psi_1 \rangle$  and  $\langle \Psi_2 | \tilde{S}(t_2, t_1) | \Psi_1 \rangle$  for describing the "physical" processes and the GDE in the form (3) keeping in mind that the contribution from the above terms in this equation must not be taken into account.

It is extremely important that, being the representation of the general quantum mechanical rules, Eqs. (2) and (3) allow one to obtain detailed information about physical processes without specifying the interaction operator. Such information can be regarded as a direct consequence of the first principles. Let us now investigate, in this way, the bound-state QED.

Expression (2) for  $\langle \Psi_2 | U(t, 0) | \Psi_1 \rangle$  in the Schrödinger picture can be rewritten in the form

$$\langle \Psi_2 | U_S(t, 0) | \Psi_1 \rangle = \frac{1}{2\pi} \int_{-\infty}^{\infty} dE e^{-iEt} \langle \Psi_2 | G(E + i0) | \Psi_1 \rangle, \quad (6)$$

where

$$G(z) = G_0(z) + G_0(z)T(z)G_0(z) \quad (7)$$

with  $G_0(z) = (z - H_0)^{-1}$  and

$$T(z) = i \int_0^{\infty} d(t_2 - t_1) \exp[iz(t_2 - t_1)] \exp[-iH_0 t_2] \tilde{S}(t_2, t_1) \exp[iH_0 t_1] \quad (8)$$

In terms of the T operator defined by Eq. (8) the generalized dynamical equation (3) can be rewritten in the form [9]

$$\frac{d \langle \psi_2 | T(z) | \psi_1 \rangle}{dz} = - \sum_n \frac{\langle \psi_2 | T(z) | n \rangle \langle n | T(z) | \psi_1 \rangle}{(z - E_n)^2} \quad (9)$$

where  $n$  stands for the entire set of discrete and continuous variables that characterize the system in full, and  $|n\rangle$  are the eigenvectors of  $H_0$ . Correspondingly, the boundary condition (4) takes the form

$$\langle \psi_2 | T(z) | \psi_1 \rangle \xrightarrow{|z| \rightarrow \infty} \langle \psi_2 | B(z) | \psi_1 \rangle, \quad (10)$$

$$B(z) = i \int_0^\infty d\tau \exp(iz\tau) H_{int}^{(s)}(\tau), \quad (11)$$

where

$$H_{int}^{(s)}(\tau) = \exp(-iH_0 t_2) H_{int}(t_2, t_1) \exp(iH_0 t_1) \quad (12)$$

is the interaction operator in the Schrödinger picture. Equation (6) coincides in its form with the standard expression that relates the evolution operator to the Green operator. In the canonical formalism the Green operator is defined as the resolvent of the total Hamiltonian  $H$ :

$$G(z) = (z - H)^{-1} \quad (13)$$

and Eq. (7) is the definition of the T-matrix. On the other hand, Eq. (6) is simply one of the forms of Eq. (2) representing the superposition principle, where the Green operator  $G(z)$  which in general can be defined as the inverse Fourier transform of the evolution operator, given by Eq. (6), can be represented in the form (7). The Green operator defined in this way coincides with  $G(z)$  defined by Eq. (13) only in the case when the interaction in the system is instantaneous, i.e., in the case when the GDE (3) is reduced to the Schrödinger equation. This definition of the Green operator is valid even in the case when the total Hamiltonian  $H$  makes no sense. This is very important because the Green operator  $G(z)$  characterizes the evolution of a system, and, in particular, the positions of its poles determine the energies of stationary states, and this makes room for solving the bound-state problem in QED in a consistent way.

## 2. The bound-state QED and the Green operator

In solving the bound state problem in QED it is convenient to include the Coulomb field into the free Hamiltonian from the very beginning. This leads us to the Furry picture in which the eigenstates  $|n\rangle$  of the Dirac-Coulomb Hamiltonian  $H_0^D$  ( $H_0^D |n\rangle = E_n^{(0)} |n\rangle$ ) are used as "free" states. The one-electron states  $|\Psi_n^{(0)}\rangle$  corresponding to the discrete spectrum of energy are just the "bare" states of a one-electron atom, i.e. the atomic states, when the atom does not

interact with the vacuum. The real atomic states are results of dressing the bare states  $|\Psi_n^{(0)}\rangle$  by this interaction. As we show below, the GDE provides a new very effective way of solving this problem.

In QED the operator  $T(z)$  describes not only the interaction between particles but also their self-action. This problem can be overcome by reduction, which amounts to the propagator  $G_0(z)$  describing the evolution of free particles being replaced by the propagator  $\tilde{G}_0(z)$  describing the evolution of particles interacting with the vacuum and, accordingly  $T(z)$  being replaced by  $\tilde{T}(z)$ , which describes particle interaction proper. Redefined in this way the free Green operator  $G_0(z)$  describes the evolution of the system in the case when the particles move freely or interact only with the vacuum. In this way the Green operator can be rewritten in the form

$$G(z) = \tilde{G}_0(z) + \tilde{G}_0(z)M(z)\tilde{G}_0(z) \quad (14)$$

From Eq. (14) it follows that the "free" Green operator  $\tilde{G}_0(z)$  should be of the form

$$\tilde{G}_0(z) = \sum_m \frac{|m\rangle\langle m|}{z - E_m^{(0)} - C_m(z)} \quad (15)$$

and the GDE can be rewritten in the terms of  $C_m(z)$  and the operator  $M(z)$ :

$$\langle m' | m \rangle \frac{dC_m(z)}{dz} = - \langle m' | P_m M(z) \left( \tilde{G}_0(z) \right)^2 M(z) | m \rangle, \quad (16)$$

$$\begin{aligned} \frac{d\langle m' | M(z) | m \rangle}{dz} &= - \langle m' | P_m^\perp M(z) \left( \tilde{G}_0(z) \right)^2 M(z) | m \rangle - \\ &- \langle m' | P_m^\perp \frac{dC_{m'}(z)}{dz} \tilde{G}_0(z) M(z) | m \rangle - \langle m' | P_m^\perp M(z) \tilde{G}_0(z) \frac{dC_m(z)}{dz} | m \rangle, \end{aligned} \quad (17)$$

where  $P_n$  is the projection operator on the state  $|\Psi_n^{(0)}\rangle$  and  $P_n + P_n^\perp = \mathbf{1}$ . The corresponding boundary conditions are as follows

$$\langle m' | M(z) | m \rangle \xrightarrow{|z| \rightarrow \infty} \langle m' | P_m^\perp B(z) | m \rangle, \quad (18)$$

$$\langle m' | m \rangle C_m(z) \xrightarrow{|z| \rightarrow \infty} \langle m' | P_m B(z) | m \rangle, \quad (19)$$

Now we can obtain the vector of the corresponding atomic state. The stationarity of the bound state of an electron in the Coulomb field  $|\Psi_n(t)\rangle$  with the energy  $E_n$  must manifest itself in the fact that the Green operator  $G(z)$  has a pole at  $z = E_n$  and a residue being the projection operator on this state:

$$G(z) = \frac{|\Psi_n\rangle\langle\Psi_n|}{z - E_n} + O(1), \quad z \rightarrow E_n \quad (20)$$

The positions of the poles of the Green operator of the form (14) are determined by the positions of the poles of the operator  $\tilde{G}_0(z)$  that in turn are determined by the equation

$$E_n - E_n^{(0)} - C_n(E_n) = 0. \quad (21)$$



where  $n$  corresponds to the discrete spectrum. For the positions of these poles to determine the atomic states  $|\Psi_n\rangle$ , their residues should be the projections on these states,  $|\Psi_n\rangle\langle\Psi_n|$ . In order to find these residues, one has to analyze all of the contributions to the Green operator having such a pole. Besides the operator  $\tilde{G}_0(z)$ , this singularity obviously takes place in

$$P_n\tilde{G}_0(z)M(z) \equiv P_n\tilde{G}_0(z)M(z)P_n^\perp$$

and

$$M(z)\tilde{G}_0(z)P_n \equiv P_n^\perp\tilde{G}_0(z)M(z)P_n.$$

Note that  $\langle\Psi_n^{(0)}|M(z)|\Psi\rangle = \langle\Psi_n^{(0)}|M(z)P_n^\perp|\Psi\rangle$  and  $\langle\Psi|M(z)|\Psi_n^{(0)}\rangle = \langle\Psi|P_n^\perp M(z)|\Psi_n^{(0)}\rangle$  by the definition. The singularity in the part of the Green operator given by  $P_n^\perp\tilde{G}_0(z)M(z)\tilde{G}_0(z)P_n^\perp$  can arise only because of the singularity of the operator  $P_n^\perp M(z)P_n^\perp$ . It is very important that Eqs. (16) and (17) allow to determine the behavior of the operator  $T(z)$  in the vicinity of the singular point  $z = E_n$  without specifying the interaction operator, and this behavior is as follows

$$M(z) = \frac{C_0}{(z-E_n)} \left\{ M(E_n)|\Psi_n^{(0)}\rangle\langle\Psi_n^{(0)}| M(E_n) + P_n^\perp M(z)P_n + P_n M(z)P_n^\perp \right\} + O(1), \quad z \rightarrow E_n \quad (22)$$

where  $C_0 = \left(1 - \frac{dC_m(z)}{dz}\bigg|_{z=E_n}\right)^{-1/2}$

The insertion of this expression in Eq. (14) yields

$$G(z) = \frac{|\Psi'_n\rangle\langle\Psi_n|}{z-E_n} + O(1), \quad z \rightarrow E_n \quad (23)$$

with

$$|\Psi_n\rangle = \overline{C}_0 \left( |\Psi_n^{(0)}\rangle + P_n^\perp \tilde{G}_0^+(E_n) M^+(E_n) |\Psi_n^{(0)}\rangle \right) \quad (24)$$

$$|\Psi'_n\rangle = C_0 \left( |\Psi_n^{(0)}\rangle + P_n^\perp \tilde{G}_0(E_n) M(E_n) |\Psi_n^{(0)}\rangle \right) \quad (25)$$

At  $z = E_1$  ( $n = 1$  corresponds to the ground state of the atom) the operators  $\tilde{G}_0(z)$  and  $M(z)$  are Hermitian and, as a result,  $|\Psi'_1\rangle = |\Psi_1\rangle$ . From this and the fact that the vector  $|\Psi_1\rangle$  is normalized  $\langle\Psi'_1|\Psi_1\rangle = 1$ , and Eq. (23), it follows that the ground state  $|\Psi_1(t)\rangle$  is stationary:  $|\Psi_1(t)\rangle = e^{-iE_1 t} |\Psi_1(0)\rangle$ . The difference  $\delta E_1^{(0)} = C(E_1)$  between the energy  $E_1$  of this state and  $E_1^{(0)}$  is caused by the interaction of the atom with the vacuum, and is just the Lamb shift of the energy level of the ground state. The vector  $|\Psi_1\rangle$  describes the ground state of the atom. This is not a one-electron state even in the sense of the Dirac equation. Actually in this case we deal with the state of the atom surrounded by a cloud of virtual particles. The second term in expression (24) for the state  $|\Psi_1\rangle$  just describes the probability to find the virtual particles such as, for example, photons and electron-positron pairs in a measurement, if before the experiment the atomic system was in the state  $|\Psi_1\rangle$ .

The excited states  $|\Psi_n(t)\rangle$  ( $n = 2, 3, \dots$ ) can be regarded as quasistationary states because their energies determined by Eq. (21) are complex  $E_n = E_n^{(0)} + C_n(E_n) \equiv E_n^{(0)} + \delta E_n - i\Gamma_n/2$ . This results in the fact that the states  $|\Psi'_n\rangle$  and  $|\Psi_n\rangle$  given by Eqs. (24) and (25) respectively do not coincide exactly.

Our analysis of the unstable atomic states can be extended to the general description in terms of the energy distribution [14]. However, in this work we will restrict ourselves to the approximation that is proven to be applicable for solving the most of the problems in quasistationary atomic physics. In this approximation the difference between the vectors  $|\Psi'_n\rangle$  and  $|\Psi_n\rangle$  can be neglected.

Formulae (21) and (24) determining the energies  $E_n$  and the vectors  $|\Psi_n\rangle$  of the atom have been derived as an inevitable consequence of Eqs. (16) and (17), which in turn represent the basic quantum mechanical rules, without resorting to any specific information about the character of the electromagnetic interaction. In order to use these formulae for the calculations one has to obtain, at first, the operator  $M(z)$  and the function  $C_n(z)$  by solving Eqs. (16) and (17). Only at this stage of our analysis we have to specify the interaction operator. The interaction operator  $B(z)$  that describes the local electromagnetic interaction is of the form

$$B(z) = H_I = \int d^3x \mathcal{H}_I(0, \mathbf{x}) = e \int d^3x : A_\mu(0, \mathbf{x}) \bar{\Psi}(0, \mathbf{x}) \gamma^\mu \Psi(0, \mathbf{x}) : \quad (26)$$

where  $\mathcal{H}_I(x)$  is the interaction Hamiltonian density,  $\Psi(t, \mathbf{x})$  is the Dirac field in the Furry picture and  $A^\mu(x)$  is the electromagnetic field. The value of the coupling constant of the electromagnetic interaction allows to solve Eqs. (16) and (17) perturbatively. In the first order of this solution we have

$$M^{(1)}(z) = H_I, \quad \tilde{G}_0^{(1)}(z) = G_0(z) \quad (27)$$

In the next order, for  $C_n(z)$  we have the equation

$$\frac{dC_n^{(2)}(z)}{dz} = - \langle \Psi_n^{(0)} | H_I (G_0(z))^2 H_I | \Psi_n^{(0)} \rangle \quad (28)$$

The form of the interaction operator (26) implies that  $C_n(z)$  tends to zero as  $|z| \rightarrow \infty$ . Solving Eq. (28) with this boundary condition yields

$$\begin{aligned} C_n^{(2)}(E) &= \int_0^\infty e^{iE_n(t_2-t_1)} C_n(t_2-t_1) d(t_2-t_1) = \int_0^\infty d(t_2-t_1) \int d^3\mathbf{p}_1 \int d^3\mathbf{p}_2 \bar{\Psi}_n(\mathbf{p}_2) \times \\ &\times \left[ \Sigma_A(\mathbf{p}_2, t_2; \mathbf{p}_1, t_1) e^{iE_n(t_2-t_1)} + \Sigma_A(\mathbf{p}_2, t_1; \mathbf{p}_1, t_2) e^{i\{(E-2E_n^{(0)})(t_2-t_1)\}} \right] \Psi_n(\mathbf{p}_1) \end{aligned} \quad (29)$$

where

$$\Sigma_A(\mathbf{p}_2, t_2; \mathbf{p}_1, t_1) = \int d^4x_1 \int d^4x_2 \Sigma_A(x_2, x_1) e^{i\mathbf{p}_2 \mathbf{x}_2} e^{-i\mathbf{p}_1 \mathbf{x}_1} \quad (30)$$

with  $\Sigma_A(x_2, x_1)$  being the ordinary one-loop self-energy operator of the electron. The leading order  $C_n(E)$  obtained in this way can then be used for obtaining

the leading order energy shift. For this one has to solve Eq. (21) which in this case is reduced to the equation

$$E - E_n^{(0)} - C_n^{(2)}(E + i0) = 0 \quad (31)$$

Neglecting the dependence of  $C_n^{(0)}(z)$  on  $z$  in the vicinity of the point  $z = E_n^{(0)}$ , from this equation for the leading order self energy correction  $\delta E_n = E - E_n^{(0)}$ , we get

$$\delta^{(0)} E_n = C_n^{(2)}(E_n^{(0)} + i0) = \int d^3 \mathbf{p}_1 \int d^3 \mathbf{p}_2 \bar{\Psi}_n(\mathbf{p}_2) \Sigma_A(\mathbf{p}_2, \mathbf{p}_1, E_n^{(0)}) \Psi_n(\mathbf{p}_1) \quad (32)$$

Here we have taken into account that  $\Sigma_A(\mathbf{p}_2, t_2; \mathbf{p}_1, t_1)$  depends on the difference  $(t_2 - t_1)$  only. Of course, the one-loop self-energy operator  $\Sigma_A(\mathbf{p}_2, \mathbf{p}_1, E_n^{(0)})$  makes no mathematical sense because of the UV divergences. But the renormalization theory is applicable in this case, and the problem is solved by replacing  $\Sigma_A(\mathbf{p}_2, \mathbf{p}_1, E_n^{(0)})$  in Eq. (32) by its renormalized value  $\Sigma_A^{(R)}(\mathbf{p}_2, \mathbf{p}_1, E_n^{(0)})$ . Eq. (32) coincides with the well known expression for the one-loop energy shift in hydrogen that in the standard bound state QED is derived, for example, from the solution of the effective Dirac equation with the mass operator [15]

$$H_0^D \Psi(x) + i \int \gamma_0 \Sigma_A(x, x', E) \Psi(x') dx' = E \Psi(x) \quad (33)$$

where  $H_0^D$  is the Dirac Hamiltonian without the mass operator but with the external field. In fact, in the leading order of the expansion of the solution of Eq. (33) in powers of  $\alpha$  we can put  $\Psi(x) = \Psi_0(x)$ , and  $\Sigma_A(\mathbf{x}, \mathbf{x}', E) = \Sigma_A(\mathbf{x}, \mathbf{x}', E_0)$  in this equation, and in this way we arrive at the expression (32) for the self-energy correction. However, (32) is not an exact one-loop energy shift, because  $E = E_n^{(0)} + C_n^{(0)}(E_n)$  is only an approximate solution of Eq. (21). At the next order of the iterative solution of this equation we get  $z = E_n^{(0)} + C_n^{(2)}(E'_n)$ . By using Eq. (29) and the representation

$$\Sigma_A(\mathbf{x}_2, \mathbf{x}_1; t_2, t_1) = \frac{1}{2\pi} \int dE e^{-iE(t_2 - t_1)} \Sigma_A(\mathbf{x}_2, \mathbf{x}_1; E) \quad (34)$$

at this order for the one-loop correction, we get

$$\delta E_n^{(2)} = \int d^3 \mathbf{p}_1 \int d^3 \mathbf{p}_2 \bar{\Psi}_n(\mathbf{p}_2) \Sigma_A(\mathbf{p}_2, \mathbf{p}_1, E'_n) \Psi_n(\mathbf{p}_1) + \delta^D E_n, \quad (35)$$

where

$$\delta^D E_n = \int d^3 \mathbf{p}_1 \int d^3 \mathbf{p}_2 \bar{\Psi}_n(\mathbf{p}_2) \Sigma^{(n)}(\mathbf{p}_2, \mathbf{p}_1) \Psi_n(\mathbf{p}_1) \quad (36)$$

with

$$\Sigma^{(n)}(\mathbf{p}_2, \mathbf{p}_1) = -\frac{\delta_n^{(0)}}{2\pi i} \int dE \frac{\Sigma_A(\mathbf{p}_2, \mathbf{p}_1, E)}{(E_n^{(0)} - E + i0)^2} \quad (37)$$

The first term on the right-hand part of Eq. (35) is the energy shift, which is determined by Eq. (32), when in this equation we put  $\Sigma_A(\mathbf{x}, \mathbf{x}', E) = \Sigma_A(\mathbf{x}, \mathbf{x}', E_0)$ . Thus, there is a discrepancy between the predictions of the standard bound state QED and our approach.

The discrepancy is the manifestation of the fact that actually there is no one-to-one correspondence between the poles of the Green operator and the poles of the respective Green functions: Eq. (34) determines the poles of the Green function of the electron in the Coulomb field. To explain this point note that the electronic Green function in the Furry picture is determined as

$$G_{ik}(x, x_0) = i \langle 0 | T \Psi_i^{(e)}(x) \bar{\Psi}_k^{(e)}(x_0) S | 0 \rangle \quad (38)$$

where  $S$  is the scattering matrix and  $\Psi_i^{(e)}(x)$  and  $\Psi_k^{(e)}(x_0)$  are the electronic field operators in the Furry picture, and  $i, j$  are spin indexes, in the one-loop approximation under study, the  $S$  matrix is given by the first two terms in the Feynman-Dyson expansion, and, as a result, the Green function takes the form

$$G_{ik}(x, x_0) = G_{ik}^{(2)}(t, \mathbf{x}; t_0, \mathbf{x}_0) = i \langle 0 | T \Psi_i(t, \mathbf{x}) \bar{\Psi}_k(t_0, \mathbf{x}_0) | 0 \rangle - \frac{i}{2} \int_{-\infty}^{\infty} d^4 x_1 \int_{-\infty}^{\infty} d^4 x_2 \langle 0 | T \Psi_i(t, \mathbf{x}) \bar{\Psi}_k(t_0, \mathbf{x}_0) \mathcal{H}_I(x_1) \mathcal{H}_I(x_2) | 0 \rangle \quad (39)$$

On the other hand, the one-electron matrix element of the evolution operator  $\langle \Psi_n^{(0)} | U(t, t_0) | \Psi_n^{(0)} \rangle$  in this approximation reads

$$\langle \Psi_n | U(t, t_0) | \Psi_n \rangle = 1 - \frac{i}{2} \int_{t_0}^t dt_1 \int_{-\infty}^{\infty} d^3 x_1 \int_{t_0}^t dt_2 \int_{-\infty}^{\infty} d^3 x_2 \langle \Psi_n | T \mathcal{H}_I(x_1) \mathcal{H}_I(x_2) | \Psi_n \rangle \quad (40)$$

Here as well as in Eq.(39) the contributions from the terms associated with the vacuum-vacuum transition should be removed. Thus, in order for the electronic Green function could be reduced to the one-electron matrix elements of the evolution operator, the integration in Eq.(39) must be limited to the time intervals  $t_0 \leq t_1 \leq t$  and  $t_0 \leq t_2 \leq t$ . This proves that the positions of the poles of the electronic Green function do not coincide exactly with the positions of the poles of the Green operator, and hence do not determine exactly the atomic energy levels. Nevertheless, for atomic hydrogen the contribution from the virtual processes that should be removed by the above limitation is relatively small, and this is a reason why the standard bound-state QED provides high accuracy in calculating the Lamb shift of the atomic energy levels. But this accuracy may turn out to be not sufficient for the derivation of the proton radius.

The above shows that actually there is a correction  $\delta^D E$  to the radiative Lamb shift in hydrogen that is hidden in solving the problem in a standard way. But the integral in Eq. (36) that determines this correction in the one-loop approximation diverges at infinity, because the one-loop self-energy operator  $\Sigma_A(\mathbf{p}', \mathbf{p}; E)$  behaves in the limit  $|E| \rightarrow \infty$  as

$$\Sigma_A(\mathbf{p}', \mathbf{p}; E) \Big|_{|z| \rightarrow \infty} \rightarrow \frac{\alpha}{4\pi} \gamma_0 E \ln \left( \frac{m^2 - (E + i0)^2}{m^2} \right) \quad (41)$$

The problem is that this divergence occurs after renormalization.

It should be noted, the above results can be trivially extended to the description of QED corrections to energy levels of muonic hydrogen.

### 3. Nonlocality of the electromagnetic interaction

We have shown that the standard methods based on the determination of the atomic levels by the positions of the poles of the Green function of the electron in the Coulomb field do not obey the exact description of the atomic states. The dynamical shift  $\delta^D E$  which in the one-loop approximation is given by Eq. (36) describes the part of the Lamb shift that was missing in the standard description and might be the origin of the discrepancy between the radii derived from atomic hydrogen and muonic hydrogen spectroscopy. At the same time, in describing this dynamical shift we face the problem of the UV divergences that cannot be cured by renormalization. In other words, the renormalization theory fails in this case. This is not surprising because in dealing with the Green operator constructed by using the renormalization procedure we are going beyond the domain of applicability of the standard theory of QED, which is limited to deal only with the process that can be described in terms of the S matrix or the Green functions. The "rug" has turned out to be too small to hide the problem of the UV divergences that arise in the consistent description of the bound states based on the employment of the Green operator. Actually, in this case the natural nonlocality of the electromagnetic interaction, which in describing processes associated with the S matrix is hidden in the renormalization procedures, manifests itself explicitly. This means, that for the accurate calculations of the QED corrections and, hence, for the accurate derivation of the proton radius from spectroscopy of atomic hydrogen and muonic hydrogen, one has to deduce the nonlocal interaction operator describing the fundamental electromagnetic interaction. Keeping in mind that taking into account the nonlocality of the electromagnetic interaction which must result in improving the behaviour of the self-energy operator at infinity, we can make some assumptions about the value of the correction  $\delta^D E$ . Formally the order of magnitude of this correction is given by  $\delta^D E_n = \alpha^3 A_0 \delta^{(0)} E_n$ , where  $\delta^{(0)} E_n$  coincides with the ordinary one-loop Lamb shift. The constant  $A_0$  should be determined by the form of the self-energy operator. Thus the nonlocality of the electromagnetic interaction must manifest itself in this constant and, hence, in the Lamb shift of atomic energy levels.

It might seem that there is a significant arbitrariness in choosing the form of the interaction operator. However, the fact that the interaction operator  $H_{int}(t_2, t_1)$  in the limit  $t_2 \rightarrow t_1$  should be close enough to the relevant solution of the GDE, imposes strong limitations on its form. In addition, the form of the interaction operator is constrained by the symmetries of QED and the fact that in describing the processes associated with the S matrix the GDE should lead to the same results that in the standard theory of QED are obtained by using the renormalization theory. A transparent example of this feature of the GDE is given in Ref. [16], where it has been shown that after renormalization

in the effective field theory of nuclear forces the low energy nucleon dynamics is governed by the GDE with a nonlocal-in-time interaction operator whose form is determined by the above constraints up to a constant that is fixed by fitting to the nucleon scattering data. Here we are not in position to discuss the problem of the determination of the form of the nonlocal interaction operator describing the fundamental electromagnetic interaction in detail. But note, that there is a hope the above limitations on the form of this operator will allow one to determine it up to a constant the value of which could be derived from the spectroscopy of atomic hydrogen and muonic hydrogen. And just the discrepancy between the radii derived from these measurements may provide us experimental information about the nonlocal nature of the electromagnetic interaction.

#### 4. Outlook

We have shown that the GDE provides a consistent way of solving the bound-state problem in QED. In this way the energies and vectors of the atomic states are determined by the positions and residues of the Green operator defined by the inverse Fourier transform (6) of the evolution operator. From Eq. (23) it immediately follows that such states are stationary. Formulae (21) and (24) derived in this way obey the accurate determination of the energies and the vectors of the atomic states. This is because they are derived as an inevitable consequence of Eqs. (2) and (3) that represent the basic quantum mechanical rules without entering into the details of the interaction in the system. As we have shown the standard bound-state QED does not obey the rigorous description of the atomic states, because strictly speaking the problem of finding the poles and residues of the Green operator is not reduced to the problem of finding the poles of the Green function of the electron in the Coulomb field.

The discrepancy between the proton radii deduced from atomic hydrogen spectroscopy and muonic hydrogen spectroscopy seems to be so serious that it could force revisions of the fundamentals of physics. However, a small part of what is allowed by the current fundamental principles of physics has been realized in existing theories. In fact these principles, as we have seen, allow nonlocal-in-time interaction operators as generators of quantum dynamics, while in the existing theories one restricts oneself to the instantaneous interaction operators which generate the Hamiltonian dynamics. And the UV divergences are just the cost for this restriction. This means that local interactions are actually inconsistent with the current physical principles. Thus in order to realize all the possibilities allowed by the current principles of physics, one has to take into account the nonlocality of the fundamental interactions explicitly, and the GDE provides a consistent way to succeed in this object. And what is important in this connection is that Eq. (2) and the GDE separated from boundary condition (4) are simply relations that represent the basic quantum mechanical rules. And these rules impose such strong limitations on the physical processes that Eq. (3) turns out to be the rule prescribing how to build the probability amplitudes of physical processes from the "bricks" associated with the contributions

from the processes with infinitesimal duration times of interaction. This rule is general and is independent of physics with which we deal. The specific physical content is introduced by choosing these "bricks", i.e. specifying the form of the fundamental interaction operator. In addition this interaction operator must obey the relativity and other symmetries of the theory.

If the theory of QED were defined in this way, then it would be finite and could provide the accurate calculations of atomic energy levels and, hence, the accurate derivation of the proton radius from the hydrogen atom and muonic atom spectroscopy. However, additional experimental information is needed to specify the nonlocal interaction operator. And the work of Pohl and colleagues may provide this information, which being reanalyzed in this way may reveal unknown features of the electromagnetic interaction. And these features may turn out to be very surprising.

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